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An Efficient Numerical Quadrature Scheme Based on Three-Point Chebyshev Nodes: An Analogy to Simpson's $\frac{1}{3}$ Rule

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Abstract

This paper presents an efficient numerical quadrature rule derived using three-point Chebyshev nodes**,** offering an analogy to Simpson's $1/3$ rule. The proposed scheme combines computational simplicity with enhanced accuracy through the use of non-equidistant nodes, which effectively minimize the Runge phenomenon commonly encountered in polynomial interpolation with equally spaced nodes. The resulting integration formula achieves a convergence order of $O(h^4)$, comparable to Simpson's $1/3$ rule, but with better error control. This approach significantly improves both the stability and accuracy of numerical integration. The method is validated against exact analytical results and benchmarked with classical quadrature rules such as the trapezoidal rule**,** Simpson's $1/3$ rule, and Simpson's $3/8$ rule**.** A composite formulation is implemented to extend the method over larger intervals. Numerical experiments demonstrate that the proposed Chebyshev-based scheme achieves superior accuracy and efficiency even with a relatively low-order formulation, emphasizing its potential for a wide range of scientific and engineering applications.

Keywords: *Chebyshev nodes; numerical quadrature; trapezoidal rule; Simpson's rule; Runge phenomenon.*

1. Introduction

In many real-world problems, evaluating a definite integral is an essential step in understanding physical, biological, or economic systems. Examples include finding the area under a curve, calculating the total quantity from a varying rate (such as velocity or current), or determining probabilities in continuous distributions. However, most integrals cannot be solved exactly, especially when the function is complicated, discontinuous, or known only at specific points. In such cases, numerical integration becomes a practical and necessary approach.

Numerical integration, also known as quadrature, is the process of estimating the value of a definite integral using numerical techniques. Over time, many methods have been developed from simple one like the trapezoidal rule to more advanced approaches such as Gaussian quadrature and adaptive integration. However, these methods involve a trade-off between accuracy, simplicity, and computational cost.

Among these, methods based on orthogonal polynomials, especially Chebyshev polynomials have attracted significant attention. Chebyshev polynomials possess unique mathematical properties that make them highly effective for approximation. Their roots (called Chebyshev nodes) minimize interpolation errors and reduce the large oscillations that often occur with equally spaced points, a problem commonly referred to as the Runge phenomenon.

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In this study, we focus on developing a three-point Chebyshev nodes-based quadrature rule that provides an analogy to Simpson's **1/3** rule. Although much of the literature discusses higher-order or generalized Chebyshev quadrature methods, our aim is to show that even a low-order formulation can produce accurate and efficient results. The three-point formulation is analogous to Simpson's 1/3 rule and computationally efficient, making it ideal for making it ideal for practical computations and applications with limited computational resources.

Through theoretical derivation and numerical testing, we demonstrate that the proposed three-point Chebyshev nodes integration scheme provides a well-balanced combination of accuracy and efficiency. Finally, its performance is evaluated against classical quadrature methods and illustrated using a variety of test functions.

2. Literature Review

Numerical quadrature schemes have been extensively developed to approximate definite integrals when analytic evaluation is impractical. Classical Newton–Cotes formulas, including the trapezoidal rule, Simpson's 1/3 rule, and Simpson's 3/8 rule, rely on polynomial interpolation at uniformly spaced nodes. Although widely used due to their simplicity, these methods generally suffer from limited accuracy when the integrand exhibits rapid variation, endpoint singularities, or oscillatory behavior.

This challenge has driven the development of alternative quadrature schemes, including those based on Chebyshev nodes, which offer distinct advantages in certain scenarios (Gilboa & Peled, 2017). Chebyshev polynomials offer several advantages for numerical approximation due to their minimax property and optimal distribution of nodes, which significantly reduce interpolation error (Mason, 1993).

Several studies have explored Chebyshev or some other quadrature schemes to enhance error performance. Esclahchi et al. (2005) developed Chebyshev–Newton–Cotes rules, demonstrating notable improvements over traditional formulas. Kythe & Schäferkotter (2004) provide practical methods for various quadrature rules, including those based on Chebyshev polynomials. They discuss the balance between computational cost and accuracy, which is a central theme of the proposed work. Burg and Degny (2013) proposed a derivative-based midpoint quadrature rule with enhanced local precision. Peer (2015) introduced geometric insights into tangent-based approximations that indirectly contribute to improved integral estimation. Ullah (2015) proposed a generalized rule with better numerical efficiency in selected computational scenarios. Collectively, these studies indicate a consistent research trend toward formulating quadrature rules that achieve a higher balance between accuracy and computational simplicity.

However, many existing Chebyshev-based approaches either rely on higher-order constructions with increased computational cost or lack direct analogies to standard Newton–Cotes rules commonly used in practice. To bridge this gap, the present work proposes a three-point Chebyshev quadrature rule that is structurally analogous to Simpson's 1/3 rule but leverages non-equidistant Chebyshev nodes. This formulation aims to maintain the simplicity and familiarity of Simpson's method while achieving superior accuracy, stability, and convergence properties.

3. Theoretical Background and Notations

3.1 Numerical Integration and Quadrature Rules

Numerical integration, also known as numerical quadrature, involves approximating the integral of a function $f(x)$ over an interval $[a, b]$ using discrete evaluations of the function at selected points.

3.2 Some Classical Newton-Cotes Quadrature Rules

3.2.1 *Trapezoidal Rule*

The Trapezoidal rule is the simplest Newton-Cotes quadrature formula, which approximates the integrand by a linear polynomial passing through two equally spaced points. For a single interval $[a, b]$, the rule is given by:

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(a) + f(b)],$$

where $h = b - a$ is the width of the single interval. This rule corresponds to the area under the straight line connecting $f(a)$ and $f(b)$, forming a trapezoid.

3.2.2 *Simpson's* $1/3$ *Rule*

The Simpson's $1/3$ rule approximates the integrand by fitting a second-degree polynomial (a parabola) through three equally spaced nodes. For a given interval $[a, b]$, the rule can be written as:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

where $h = \frac{b-a}{2}$.

This rule provides exact results for polynomials of degree three or less and is known for its higher accuracy compared to the trapezoidal rule.

3.2.3 *Simpson's* $3/8$ *Rule*

The Simpson's $3/8$ Rule uses a cubic polynomial to approximate the integrand over four equally spaced points. For a single application over the interval $[a, b]$, it is given by:

$$\int_a^b f(x)dx \approx \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)],$$

where $h = \frac{b-a}{3}$. The rule provides higher accuracy than Simpson's $1/3$ rule for certain smooth functions due to its use of a higher-degree interpolating polynomial.

3.3 Composite Rules

To improve the accuracy of numerical integration over a finite interval, composite quadrature rules are employed by subdividing the integration domain into smaller subintervals and applying the respective rule repeatedly.

Let the interval of integration $[a, b]$ be divided into n equal subintervals by defining a set of points $x_0, x_1, \dots, x_n$, where

$$x_i = a + ih, \quad i = 0, 1, \cdots n, \quad \text{and} \quad h = \frac{b-a}{n}$$

is the uniform step size.

3.3.1 *Composite Trapezoidal Rule*

In the composite trapezoidal rule, the function is evaluated at the endpoints of each subinterval to approximate the integral as sum of the Trapezoidal areas. For $n(\geq 1)$ subintervals, the composite trapezoidal rule is given by:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right].$$

3.3.2 *Composite Simpson's 1/3 Rule*

This method applies Simpson's 1/3 rule over successive pairs of subintervals. Thus, n must be even. For n subintervals, the composite Simpson's 1/3 rule is:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{(n-2)/2} f(x_{2i}) + f(x_n)\right].$$

3.3.3 *Composite Simpson's 3/8 Rule*

Simpson's 3/8 rule uses a cubic polynomial over three consecutive subintervals. Thus, n must be a multiple of 3. For n subintervals, the composite Simpson's 3/8 rule is:

$$\int_a^b \boldsymbol{f(x)\,dx} \approx \frac{\boldsymbol{3h}}{\boldsymbol{8}}\left[\boldsymbol{f(x_0) + 3\sum_{i=1}^{n/3}\left(f(x_{3i-2}) + f(x_{3i-1})\right) + 2\sum_{i=1}^{(n-3)/3} f(x_{3i}) + f(x_n)}\right].$$

3.4 Chebyshev Polynomials

3.4.1 *Chebyshev Polynomials of First Kind*

The Chebyshev polynomials of the first kind of degree n, denoted by $T_n(x)$, are defined on the interval $[-1, 1]$ as follows (Mason, J.C., 1993):

$$T_n(x) = \cos(n\theta),$$

where $\theta = \cos^{-1} x$, with $0 \leq \theta \leq \pi$ and $n = 0, 1, 2, \cdots$.

3.4.2 *Recurrence Relation of $T_n(x)$*

A three-term recurrence relation between consecutive Chebyshev polynomials of first kind in standard form is

$$T_0(x) = 1,$$

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x), \text{ for } n \geq 1.$$

3.4.3 *Nodes of $T_n(x)$*

The Chebyshev polynomial $T_n(x)$, for $n \geq 1$ has n simple, real nodes in the interval $(-1, 1)$ and they are given by

$$x_j = \cos\left(\frac{2j+1}{2n}\pi\right), \qquad j = 0,1, \ldots, n-1.$$

3.4.4 *Chebyshev Polynomials on* $[a, b]$ *and Shifted Chebyshev Polynomials*

The standard domain of the Chebyshev polynomials is $[-1, 1]$. However, it is possible to define Chebyshev polynomials over any generic interval $[a, b]$ using the technique of change of variable:

$$\bar{x} = \frac{x - \frac{1}{2}(a+b)}{\frac{1}{2}(b-a)}.$$

Thus, Chebyshev nodes $\bar{x}_j$ of the Chebyshev polynomial of the first kind of degree n on $[a, b]$ will be given by

$$\bar{x}_j = \frac{1}{2}\left[(b-a)\cos\left(\frac{2j+1}{2n}\pi\right) + a + b\right],\ j = 0,1,\dots,n-1.$$

The Chebyshev polynomials of the first kind on the interval $[0, 1]$ called Shifted Chebyshev polynomials, denoted by $T_n^*(x)$ and are defined as

$$T_n^*(x) = T_n(2x-1).$$

3.4.5 *Nodes of Third-Degree Chebyshev Polynomial and Their Properties*

The nodes of the third-degree Chebyshev polynomial $T_3(x)$ mapped onto the interval $[a, b]$ are given by:

$$z_k = \frac{1}{2}(a+b) - \frac{1}{2}(b-a)\cos\left(\frac{(2k+1)\pi}{6}\right), \quad k = 0, 1, 2.$$

Evaluating the cosine terms results in the following explicit expressions:

$$z_0 = \frac{1}{2}(a+b) - \frac{\sqrt{3}}{4}(b-a), \quad z_1 = \frac{1}{2}(a+b) \text{ and } z_2 = \frac{1}{2}(a+b) + \frac{\sqrt{3}}{4}(b-a).$$

It is worth to note that the mid-point of any interval is one of the nodes of third degree Chebyshev Polynomial in the respective interval.

3.4.5.1 *Spacing between Nodes of Third-Degree Chebyshev Polynomial*

Let $l = b - a$ denote the interval length. The distance between adjacent Chebyshev nodes is:

$$z_1 - z_0 = z_2 - z_1 = \frac{\sqrt{3}}{4}(b-a) = \mu l, \qquad \text{where } \mu = \frac{\sqrt{3}}{4}. \tag{1}$$

3.4.5.2 *Spacing from Endpoints*

The distance from the endpoints of the interval to the nearest Chebyshev nodes is:

$$z_0 - a = b - z_2 = \frac{(2-\sqrt{3})}{4}(b-a) = \lambda l, \qquad \text{where } \lambda = \frac{(2-\sqrt{3})}{4}. \tag{2}$$

From the above relationships, we observe that the nodes of the third-degree Chebyshev polynomials are symmetrically spaced about the midpoint of the interval, and the spacing depends solely on the interval length l.

4. Development of the Proposed Method

4.1 Objective

The primary objective of this study is to develop an efficient quadrature rule for approximating definite integrals of the form:

$$\int_a^b f(x)dx. \tag{3}$$

by constructing a quadratic polynomial interpolant based on the nodes of the third-degree Chebyshev polynomial.

4.2 Quadratic Interpolation using Three-Point Chebyshev Nodes

Let $P(x) = px^2 + qx + r$ be the quadratic polynomial interpolant to $f(x)$ at the nodes z_0, z_1, and z_2 of third-degree Chebyshev polynomial over the interval $[a, b]$. Then:

$$f(x) \approx P(x) = px^2 + qx + r; \quad x \in [a, b]. \tag{4}$$

Using the relationship between endpoints of the integral domain and the three-point Chebyshev nodes in the interval (see equation (2)), the definite integral of $f(x)$ over the interval $[a, b]$ can be approximated as:

$$\int_a^b f(x)\,dx \approx \int_a^b P(x)\,dx = \int_{z_0-\lambda l}^{z_2+\lambda l} (px^2 + qx + r)\,dx, \tag{5}$$

which evaluates to:

$$\int_a^b f(x)\,dx \approx \frac{p}{3}[(z_2 + \lambda l)^3 - (z_0 - \lambda l)^3] + \frac{q}{2}[(z_2 + \lambda l)^2 - (z_0 - \lambda l)^2] + r[(z_2 + \lambda l) - (z_0 - \lambda l)]. \tag{6}$$

By simplifying and regrouping the terms, the equation (6) can be expressed as:

$$\int_a^b f(x)\,dx \approx \frac{l}{6}\left[(pz_0{}^2 + qz_0 + q) + 4(pz_1{}^2 + qz_1 + r) + (pz_2{}^2 + qz_2 + r) + \frac{1}{8}pl^2\right]. \tag{7}$$

The interpolated function values at the Chebyshev nodes are:

$$\begin{aligned} f(z_0) &= pz_0{}^2 + qz_0 + r, \\ f(z_1) &= pz_1{}^2 + qz_1 + r, \\ f(z_2) &= pz_2{}^2 + qz_2 + r. \end{aligned} \tag{8}$$

Solving the system allows the coefficient p to be expressed in terms of function values at the nodes:

$$p = \frac{8}{3l^2}[f(z_0) - 2f(z_1) + f(z_2)]. \tag{9}$$

Substituting equations (8) and (9) into equation (7) gives the integral approximation entirely in terms of function values at the Chebyshev nodes:

$$\int_a^b f(x)\,dx \approx \frac{l}{9}[2f(z_0) + 5f(z_1) + 2f(z_2)]. \tag{10}$$

Expressing the Chebyshev nodes in terms of the interval endpoints using the parameter λ (equation (2)) formula (10) becomes:

$$\int_a^b f(x)\,dx \approx \frac{l}{9}\left[2f(a + \lambda l) + 5f\left(\frac{a+b}{2}\right) + 2f(b - \lambda l)\right], \tag{11}$$

where $l = b - a$ and $\lambda = \frac{(2-\sqrt{3})}{4}$.

To construct a quadrature method analogous to Simpson's 1/3 rule, we divide the integration interval into two equal subintervals, placing one Chebyshev node at the midpoint and the other two within each subinterval. Replacing l by $2h$ in equation (11), we obtain the three-point Chebyshev quadrature rule:

$$\int_a^b f(x)\,dx \approx \frac{2h}{9}[2f(z_0) + 5f(z_1) + 2f(z_2)], \tag{12}$$

or in terms of the endpoints of the integration interval

$$\int_a^b f(x)\,dx \approx \frac{2h}{9}\left[2f(a+2\lambda h) + 5f\left(\frac{a+b}{2}\right) + 2f(b-2\lambda h)\right], \tag{13}$$

where $h = \frac{b-a}{2}$ and $\lambda = \frac{(2-\sqrt{3})}{4}$.

4.3 The Composite Three-Point Chebyshev Quadrature Rule

The proposed method utilizes three Chebyshev nodes to approximate the integrand by fitting a quadratic polynomial over successive pairs of subintervals. Since the method spans two subintervals per application, the total number of subintervals n must be even.

Let the interval of integration $[a, b]$ be partitioned into n equal subintervals by introducing the grid points $x_0,\ x_1, \dots, x_n$, where

$$x_i = a + ih, \quad i = 0, 1, \cdots, n,$$

and

$$h = \frac{b-a}{n}$$

is the uniform step size.

Then, using equation (13), the composite quadrature formula for the proposed method over n equally subintervals in the interval $[a, b]$is given by:

$$\int_a^b f(x)\,dx \approx \frac{2h}{9}\left[2\sum_{i=0}^{(n-2)/2} f(x_{2i} + 2\lambda h) + 5\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2} f(x_{2i} - 2\lambda h)\right], \tag{14}$$

where $\lambda = \frac{(2-\sqrt{3})}{4}$.

4.4 Error Bound for Three-Point Chebyshev Quadrature Rule

Let integration interval $[a, b]$ be divided into n equal subintervals, with n even. Define the set of points $x_0, x_1, \dots, x_n$ as

$$x_i = a + ih, \qquad i = 0, 1, \cdots n, \qquad h = \frac{b-a}{n},$$

where h is the uniform step size.

Using the three-point Chebyshev quadrature rule (equation (12)), the approximate area $\tilde{A}$ under the parabola over the interval $[x_0, x_2]$ (the first two subintervals) is

$$\tilde{A} = \frac{2h}{9}[2f(z_0) + 5f(z_1) + 2f(z_2)], \tag{15}$$

where $h = \frac{x_2 - x_0}{2}$, and z_0, z_1, z_2 are nodes of the corresponding third-degree Chebyshev polynomial on $[x_0, x_2]$.

Consider the Taylor series expansion of $f(x)$ about $x = z_1$:

$$f(x) = f(z_1) + (x - z_1)f'(z_1) + \frac{1}{2!}(x - z_1)^2 f''(z_1) + \frac{1}{3!}(x - z_1)^3 f'''(z_1) + \cdots. \tag{16}$$

From equations (1) and (2), the Chebyshev nodes are

$$z_1 = x_1 = \frac{x_0 + x_2}{2},\ z_0 = x_0 + \lambda l = z_1 - 2\mu h,\ \ z_2 = x_2 - \lambda l = z_1 + 2\mu h,$$

where $\lambda = \frac{2 - \sqrt{3}}{4}$ and $\mu = \frac{\sqrt{3}}{4}$.

Substituting $x = z_0 = z_1 - 2\mu h$ and $x = z_2 = z_1 + 2\mu h$ into (16) we obtain

$$f(z_0) = f(z_1) - 2\mu h f'(z_1) + \frac{4}{2!}(\mu h)^2 f''(z_1) - \frac{8}{3!}(\mu h)^3 f'''(z_1) + \cdots, \tag{17}$$

and

$$f(z_2) = f(z_1) + 2\mu h f'(z_1) + \frac{4}{2!}(\mu h)^2 f''(z_1) + \frac{8}{3!}(\mu h)^3 f'''(z_1) + \cdots. \tag{18}$$

Substituting (17) and (18) into (15), $\tilde{A}$ can be written as

$$\tilde{A} = \frac{2h}{9}\left[9f(z_1) + 4\left(\frac{4}{2!}(\mu h)^2 f''(z_1) + \frac{16}{4!}(\mu h)^4 f^{(4)}(z_1) + \frac{64}{6!}(\mu h)^6 f^{(6)}(z_1) + \cdots\right)\right]. \tag{19}$$

Setting $\mu = \frac{\sqrt{3}}{4}$ and simplifying, we obtain

$$\tilde{A} = 2hf(z_1) + \frac{h^3}{3}f''(z_1) + \frac{h^5}{48}f^{(4)}(z_1) + \frac{h^7}{1920}f^{(6)}(z_1) + \cdots. \tag{20}$$

The exact area A, under $f(x)$ over $[x_0, x_2]$ is

$$A = \int_{x_0 = z_1 - h}^{x_2 = z_1 + h} \left[f(z_1) + (x - z_1)f'(z_1) + \frac{1}{2!}(x - z_1)^2 f''(z_1) + \frac{1}{3!}(x - z_1)^3 f'''(z_1) + \cdots\right] dx. \tag{21}$$

Evaluating the integral gives

$$A = 2hf(z_1) + \frac{h^3}{3}f''(z_1) + \frac{h^5}{60}f^{(4)}(z_1) + \frac{h^7}{2520}f^{(6)}(z_1) + \cdots. \tag{22}$$

Therefore, the error over $[x_0, x_2]$ is

$$\begin{aligned} A - \tilde{A} &= \left(\frac{1}{60} - \frac{1}{48}\right)h^5 f^{(4)}(z_1) + \left(\frac{1}{2520} - \frac{1}{1920}\right)h^5 f^{(6)}(z_1) + \cdots, \\ &\approx -\frac{h^5}{240}f^{(4)}(z_1) = -\frac{h^5}{240}f^{(4)}(x_1). \quad (\because z_1 \equiv x_1) \end{aligned} \tag{23}$$

Applying the same argument to each pair of subintervals, the total error E for the entire interval $[a, b]$ is

$$E = -\frac{h^5}{240}\left[f^{(4)}(x_1) + f^{(4)}(x_3) + \cdots + f^{(4)}(x_{n-1})\right].$$

Let $f^{(4)}(\xi) = \max\left\{f^{(4)}(x_1), f^{(4)}(x_3), \cdots, f^{(4)}(x_{n-1})\right\}$.

Then the maximum error $E_{max} \leq -\frac{nh^5}{480} f^{(4)}(\xi) \qquad \left(\because \text{number of interval is } \frac{n}{2}\right)$

$$= -\frac{nhh^4}{480} f^{(4)}(\xi) = -\frac{(b-a)h^4}{480} f^{(4)}(\xi) \qquad \left(\because h = \frac{b-a}{n}\right)$$

Hence the error in proposed three-point Chebyshev quadrature method is fourth order accurate, i.e., $O(h^4)$, where $h = \frac{b-a}{n}$ is the subinterval width.

5. Results and Discussion

For a fair comparison of the quadrature rules, it is essential to account for both their structural constraints such as the number of nodes (in other words number of subintervals) and their computational effort, typically measured in terms of number of function evaluations.

Accordingly, the performance of each quadrature rule is evaluated using the following two criteria:

(i) The total number of subintervals (n)

This refers to the partitioning of the integration domain into n subintervals. Each quadrature rule has condition on n, as summarized in the table below:

Table 01: Requirements on the Number of Subintervals for Different Quadrature Rules

Rule	Condition on n	Application of the Rule
Trapezoidal	Any integer	Applied over each subinterval
Simpson's 1/3	Must be even	Applied over 2 subintervals
Simpson's 3/8	Must be multiple of 3	Applied over 3 subintervals
Chebyshev	Must be even	Applied over 2 subintervals covering 3 Chebyshev nodes

Thus, for concurrent applicability of all rules, the number of subintervals n must be chosen as a common multiple of the constraints above. The least common multiple (LCM) of 2 and 3 is 6, hence suitable values include $n = 6, 12, 18, \cdots$.

(ii) Number of Panels (k)***:***

Alternatively, the integration domain can be partitioned into k panels, where each panel represents a subdomain over which a single, complete application of the respective

quadrature rule is executed. The number of discretization nodes per panel, as well as the total function evaluations, is summarized as follows:

Table 02: Comparison of Number of Function Evaluations in Different Quadrature Methods.

Rule	Nodes per Panel	Total Function Evaluations
Trapezoidal	2 (endpoints)	$k + 1$
Simpson's 1/3	3 (equally spaced)	$2k + 1$
Simpson's 3/8	4 (equally spaced)	$3k + 1$
Chebyshev	3 (Chebyshev nodes)	$3k$

Numerical Tests

In this section, we present three illustrative examples to evaluate the efficiency of the proposed Chebyshev nodes-based rule (*Cheb*) by comparing it with the trapezoidal rule (*Trap*), Simpson's 1/3 rule (*Simp13*), and Simpson's 3/8 rule (*Simp38*).

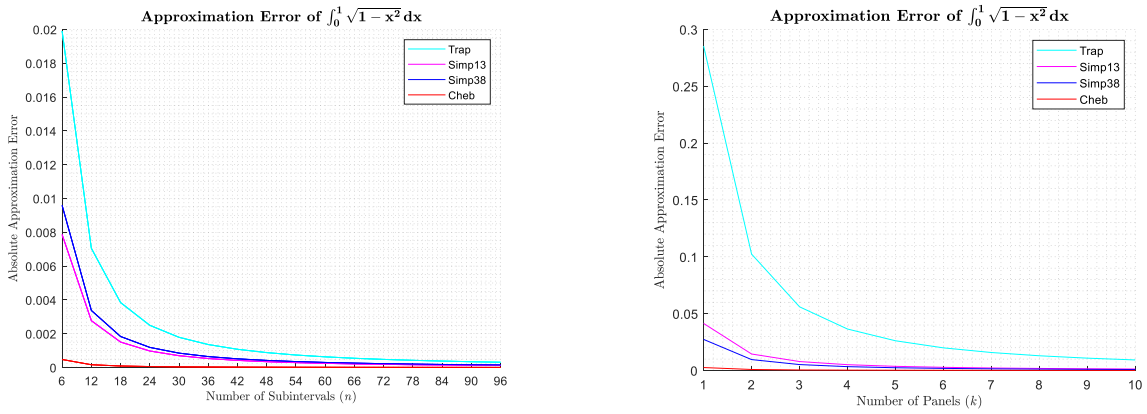
Example I: $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4} \approx 0.785398163397448.$

Table 03: Approximate Value (AV) and Absolute Approximation Error (AAE) for $\int_0^1 \sqrt{1-\mathrm{x}^2}\,\mathrm{dx}$ Using Various Quadrature Rules with Number of Subintervals $\mathrm{n} = 6, 12, 48, 96$.

	n − Number of Subintervals							
	$n = 6$		$n = 12$		$n = 48$		$n = 96$	
Rule	AV	AAE	AV	AAE	AV	AAE	AV	AAE
Trap	7.6550e-01	1.9902e-02	7.7834e-01	7.0544e-03	7.8451e-01	8.8349e-04	7.8509e-01	3.1246e-04
Simp13	7.7753e-01	7.8666e-03	7.8263e-01	2.7718e-03	7.8505e-01	3.4556e-04	7.8528e-01	1.2212e-04
Simp38	7.7581e-01	9.5920e-03	7.8202e-01	3.3774e-03	7.8498e-01	4.2081e-04	7.8525e-01	1.4870e-04
Cheb	7.8492e-01	4.7625e-04	7.8523e-01	1.6619e-04	7.8538e-01	2.0560e-05	7.8539e-01	7.2562e-06

Table 04: Approximate Value (AV) and Absolute Approximation Error (AAE) for $\int_0^1 \sqrt{1-\mathrm{x}^2}\,\mathrm{dx}$ Using Various Quadrature Rules with Number of Panels $\mathrm{k} = 1, 2, 6, 10$.

	k − Number of Panels (equivalent to number of applications of the rule)							
	$k = 1$		$k = 2$		$k = 6$		$k = 10$	
Rule	AV	AAE	AV	AAE	AV	AAE	AV	AAE
Trap	5.0000e-01	2.8540e-01	6.8301e-01	1.0239e-01	7.6550e-01	1.9902e-02	7.7613e-01	9.2686e-03
Simp13	7.4402e-01	4.1381e-02	7.7090e-01	1.4499e-02	7.8263e-01	2.7718e-03	7.8411e-01	1.2864e-03
Simp38	7.5806e-01	2.7336e-02	7.7581e-01	9.5920e-03	7.8356e-01	1.8358e-03	7.8455e-01	8.5222e-04
Cheb	7.8281e-01	2.5847e-03	7.8451e-01	8.8564e-04	7.8523e-01	1.6619e-04	7.8532e-01	7.6819e-05



(a) ***(b)***

Figure 01: Graphs of the Absolute Approximation Error for $\int_0^1 \sqrt{1-x^2}\,dx$: (a) Error vs. Number of Subintervals, and (b) Error vs. Number of Panels, for Different Quadrature Rules.

The integrand $\sqrt{1-x^2}$ used in this example is a smooth and well-behaved function over the interval $[0,1]$. As observed in Table 03 and Figure 01(a), all quadrature schemes demonstrate a consistent decrease in error as the number of subintervals n increases. However, the proposed three-point Chebyshev quadrature rule delivers significantly superior performance across every tested discretization. Notably, even at a relatively small grid size of $n = 6$, the proposed method achieves an error nearly an order of magnitude smaller than both Simpson's 1/3 and 3/8 rules.

Furthermore, as illustrated in Table 04 and Figure 01(b), increasing the number of panels (i.e., repeated applications of the rule), Chebyshev rule continues to outperform, achieving errors below 10^{-5}, thereby confirming its high accuracy for smooth functions.

Example II: $\int_{0.1}^{2.5} (3\ln x + 2x^2)\,dx \approx 10.778956016954378.$

Table 05: Approximate Value (AV) and Absolute Approximation Error (AAE) for $\int_{0.1}^{2.5}(3\ln x + 2x^2)\,dx$ Using Various Quadrature Rules with Number of Subintervals $n = 6, 12, 48, 96$.

	n − Number of Subintervals							
	$\boldsymbol{n=6}$		$\boldsymbol{n=12}$		$\boldsymbol{n=48}$		$\boldsymbol{n=96}$	
Rule	AV	AAE	AV	AAE	Value	AV	AV	AAE
Trap	1.0596e+01	1.8303e-01	1.0723e+01	5.6057e-02	1.0775e+01	3.9511e-03	1.0778e+01	9.9680e-04
Simp13	1.0700e+01	7.9412e-02	1.0765e+01	1.3734e-02	1.0779e+01	1.6198e-04	1.0779e+01	1.2032e-05
Simp38	1.0668e+01	1.1068e-01	1.0758e+01	2.0933e-02	1.0779e+01	3.1307e-04	1.0779e+01	2.5421e-05
Cheb	1.0760e+01	1.8941e-02	1.0775e+01	4.0754e-03	1.0779e+01	5.8447e-05	1.0779e+01	4.4585e-06

Table 06: Approximate Value (AV) and Absolute Approximation Error (AAE) for $\int_{0.1}^{2.5}(3\ln x + 2x^2)\,dx$ Using Various Quadrature Rules with Number of Panels $k = 1, 2, 6, 10$.

	k − Number of Panels (equivalent to number of applications of the rule)							
	$\boldsymbol{k=1}$		$\boldsymbol{k=2}$		$\boldsymbol{k=6}$		$\boldsymbol{k=10}$	
Rule	AV	AAE	AV	AAE	AV	AAE	AV	AAE
Trap	1.0033e+01	7.4562e-01	1.0017e+01	7.6177e-01	1.0596e+01	1.8303e-01	1.0701e+01	7.7579e-02
Simp13	1.0012e+01	7.6716e-01	1.0583e+01	1.9547e-01	1.0765e+01	1.3734e-02	1.0776e+01	3.0902e-03
Simp38	1.0317e+01	4.6240e-01	1.0668e+01	1.1068e-01	1.0772e+01	6.9403e-03	1.0777e+01	1.4892e-03
Cheb	1.0675e+01	1.0437e-01	1.0740e+01	3.9124e-02	1.0775e+01	4.0754e-03	1.0778e+01	1.0167e-03

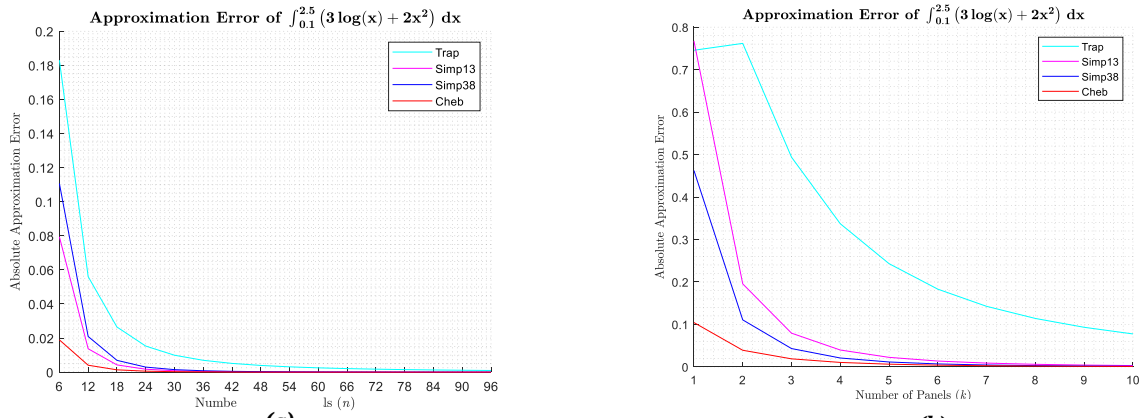


Figure 02: *Graphs of the Absolute Approximation Error for* $\int_{0.1}^{2.5}(3\ln x + 2x^2)\,dx$*: (a) Error vs. Number of Subintervals, and b) Error vs. Number of Panels, for Different Quadrature Rules.*

The integrand in this test problem, $3\ln(x) + 2x^2$ involves a logarithmic term that becomes steep near the lower limit $x = 0.1$, making the function less smooth than the one considered in Example I. As shown in Table 05, Table 06, and Figure 02(a–b), all quadrature methods initially produce relatively larger errors; however, the accuracy improves steadily as the number of subintervals increases. Among the methods considered, the proposed Chebyshev quadrature achieves the most rapid and consistent reduction in error, confirming that the proposed scheme remains robust even when the integrand exhibits mild singularity-like behavior near the boundary of the domain.

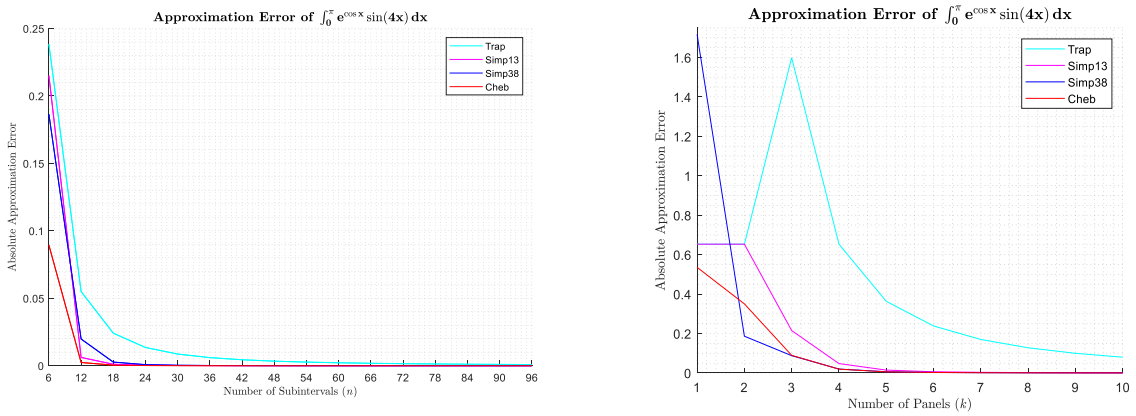
Example III: $\int_0^{\pi} e^{\cos x}\sin 4x\,dx = \frac{120}{e} - 16e \approx 0.6530236852283537.$

Table 07: Approximate Value (AV) and Absolute Approximation Error (AAE) for $\int_0^{\pi} e^{\cos x}\sin 4x\,dx$ Using Various Quadrature Rules with Number of Subintervals $n = 6, 12, 48, 96$.

	n − Number of Subintervals							
	$\boldsymbol{n = 6}$		$\boldsymbol{n = 12}$		$\boldsymbol{n = 48}$		$\boldsymbol{n = 96}$	
Rule	AV	AAE	AV	AAE	AV	AAE	AV	AAE
Trap	4.1474e-01	2.3828e-01	5.9805e-01	5.4972e-02	6.4966e-01	3.3609e-03	6.5218e-01	8.3933e-04
Simp13	8.6804e-01	2.1502e-01	6.5916e-01	6.1314e-03	6.5304e-01	1.9352e-05	6.5302e-01	1.1981e-06
Simp38	4.6658e-01	1.8644e-01	6.7281e-01	1.9784e-02	6.5307e-01	4.4102e-05	6.5303e-01	2.7043e-06
Cheb	7.4240e-01	8.9380e-02	6.5540e-01	2.3729e-03	6.5303e-01	7.2706e-06	6.5302e-01	4.4951e-07

Table 08: Approximate Value (AV) and Absolute Approximation Error (AAE) for $\int_0^{\pi} e^{\cos x}\sin 4x\,dx$ Using Various Quadrature Rules with Number of Panels $k = 1, 2, 6, 10$.

	k − Number of Panels (equivalent to number of applications of the rule)							
	$\boldsymbol{k = 1}$		$\boldsymbol{k = 2}$		$\boldsymbol{k = 6}$		$\boldsymbol{k = 10}$	
Rule	AV	AAE	AV	AAE	AV	AAE	AV	AAE
Trap	0	6.5302e-01	0	6.5302e-01	4.1474e-01	2.3828e-01	5.7301e-01	8.0018e-02
Simp13	0	6.5302e-01	0	6.5302e-01	6.5916e-01	6.1314e-03	6.5371e-01	6.8324e-04
Simp38	1.0633e+00	1.7163e+00	4.6658e-01	1.8644e-01	6.5568e-01	2.6598e-03	6.5332e-01	3.0118e-04
Cheb	1.1887e+00	5.3565e-01	3.0277e-01	3.5025e-01	6.5540e-01	2.3729e-03	6.5328e-01	2.5903e-04



(a) ***(b)***

Figure 03: *Graphs of the Absolute Approximation Error for* $\int_0^\pi e^{\cos x}\sin 4x\,dx$*: (a) Error vs. Number of Subintervals, and (b) Error vs. Number of Panels, for Different Quadrature Rules.*

The function in Example III, $e^{\cos x}\sin 4x$, is highly oscillatory over the interval $[0,\pi]$. The results in Table 07, Table 08, and Figure 03(a–b) show that for a very small number of subintervals or panels, all quadrature rules produce noticeable errors. Interestingly, for $k=1$, Simpson's 1/3 rule outperforms even Simpson's 3/8 rule, and for $k=2$, Simpson's 3/8 rule performs better than the proposed Chebyshev-based rule. This behavior can be attributed to the rapid oscillations of the integrand, which reduces the effectiveness due to the inadequacy of low-resolution polynomial interpolation in representing oscillatory functions. However, as the number of panels increases, the Chebyshev quadrature stabilizes more quickly and achieves significantly smaller errors than both the trapezoidal and Simpson's rules. This demonstrates the advantage of using Chebyshev nodes, which more effectively capture oscillatory variations in the function and improve numerical stability.

6. Conclusion

In this study, we presented an efficient three-point Chebyshev nodes-based quadrature rule constructed as an analogy to Simpson's 1/3 rule. By employing non-equidistant Chebyshev nodes, the proposed method effectively captures the behavior of the integrand and achieves higher accuracy compared to classical Newton–Cotes rules, without a significant increase in computational cost. Numerical experiments involving three different types of integrands smooth, weakly singular and highly oscillatory, clearly demonstrate the accuracy, stability, and robustness of the proposed scheme. In all cases, the proposed Chebyshev quadrature rule demonstrated faster convergence and consistently smaller approximation errors compared to the trapezoidal rule, Simpson's 1/3 rule, and Simpson's 3/8 rule. Moreover, the accuracy improved progressively as the number of subintervals increased. These findings confirm that even a low-order Chebyshev formulation can achieve an excellent balance between computational simplicity and numerical precision, making it well-suited for a wide range of scientific and engineering applications. Future work may focus on examining its performance for strongly singular integrands and other challenging scenarios, as well as extending the approach to higher-order Chebyshev-based quadrature rules.

7. References

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Appendices:

A) MATLAB Code for the Proposed Three-Point Chebyshev Quadrature Rule

```
function I = chebyshev3_quad(f, a, b, n)
%CHEBYSHEV3 Composite three-point Chebyshev quadrature
%   I = chebyshev3(f, a, b, n) approximates the integral of f from a to b
%   using n subintervals (n must be even)
%
%   Inputs:
%       f : function handle (e.g., @(x) x.^2)
%       a : lower limit of integration
%       b : upper limit of integration
%       n : number of subintervals (must be even)
%
%   Output:
%       I : approximate integral of f from a to b
    format long

    if mod(n,2) ~= 0
        error('Number of subintervals n must be even.');
    end

    h = (b - a) / n;          % subinterval width
    lambda = (2 - sqrt(3)) / 4;  % Chebyshev node parameter
    I = 0;                     % initialize integral

    % Apply the 3-point Chebyshev quadrature on each pair of subintervals
    for i = 0:2:(n-2)
        x0 = a + i*h;
        x2 = a + (i+2)*h;
        z0 = x0 + 2*lambda*h;
        z1 = (x0 + x2)/2;
        z2 = x2 - 2*lambda*h;

        % Integral over this panel
        I_panel = (2*h/9) * (2*f(z0) + 5*f(z1) + 2*f(z2));
        I = I + I_panel;
    end
end
```

B) Example 1: $\int_{0.1}^{2.5}(3\ln x + 2x^2)\,dx$ with $n = 12$ (number of subinterval).

```
>> I = chebyshev3_quad(@(x) 3*log(x)+2*x^2, 0.1, 2.5, 12)

I =

  10.774880620574198
```

C) Example 2: $\int_{0}^{1}\sqrt{1-x^2}\,dx$ with $k = 2$ (number of application of the rule).

```
>> I = chebyshev3_quad(@(x) sqrt(1-x^2), 0, 1, 4)

I =

   0.784512523002727
```

For k applications of the Chebyshev rule, the number of subintervals $n = 2k$.